Hamiltonian cycles in hypercubes with removed vertices

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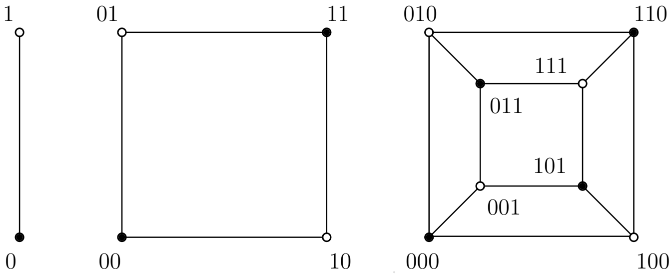
Hypercube

The n-dimensional hypercube Q_n is the graph with the vertex set $V = \{0, 1\}^n$ and two vertices are connected with an edge if and only if they differ in exactly one coordinate.



Hypercube is

- a bipartite graph
- vertex and edge transitive

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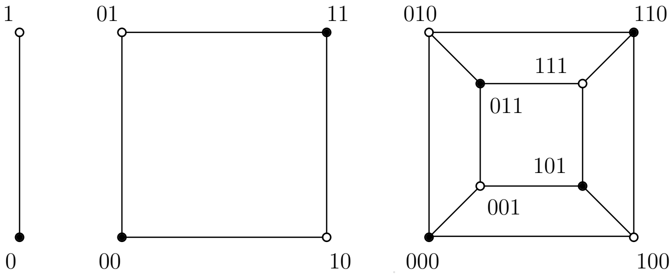

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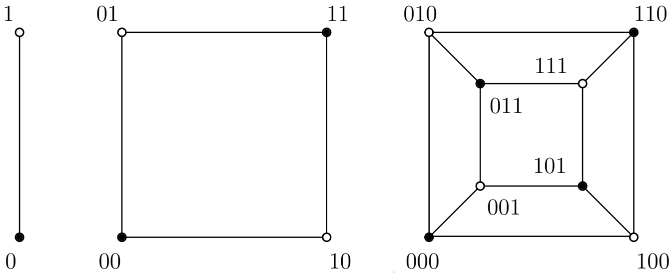

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Hamiltonicity

Let H be a bipartite graph with bipartition U, V.

We say it is:

- *balanced* if $|U| = |V|$
- *almost-balanced* if $|U|$ and $|V|$ differ by one
- *Hamiltonian laceable* if
  (a) H is balanced and there exists a Hamiltonian path between every $u \in U$ and every $v \in V$, or
  (b) H is almost-balanced with U its larger bipartite set and there exists a Hamiltonian path between every two distinct $u, u' \in U$.

Havel (1984)

The hypercube Q_n is Hamiltonian laceable for $n \geq 2$.

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Fault-tolerance

Is hypercube Hamiltonian laceble if we remove some of its vertices?

Lewinter-Widulski (1997)

For $n \geq 2$ and for $u \in V(Q_n)$ the graph $Q_n - u$ is Hamiltonian laceable.

Locke's conjecture (2003)

Hypercube Q_n with k deleted vertices ($k \geq 1$) of each parity is Hamiltonian if $n \geq k + 2$.

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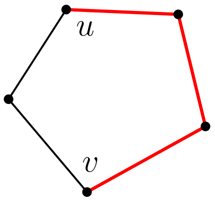
Subgraph H of G is *isometric* if it preserves all distances.



Theorem:

Let P be odd (even) isometric path in Q_n, then for $n \geq 4$ ($n \geq 5$) the graph $Q_n - V(P)$ is Hamiltonian laceable.

Sun-Joe (2009)

The hypercube Q_n without a path (or a cycle) of order at most $2n - 4$ is Hamiltonian laceable for $n \geq 4$.

Faulty subgraphs

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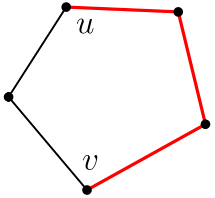

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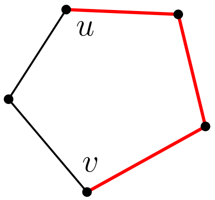

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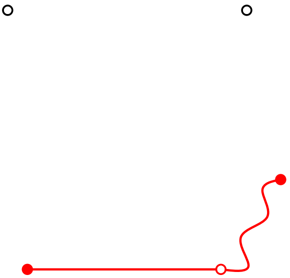

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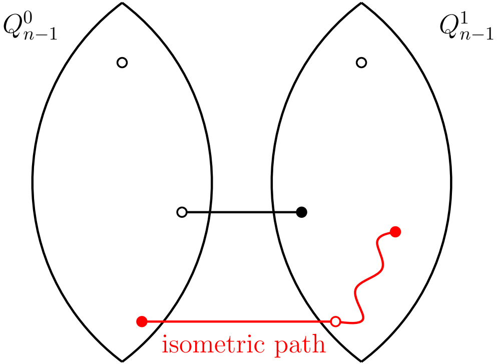

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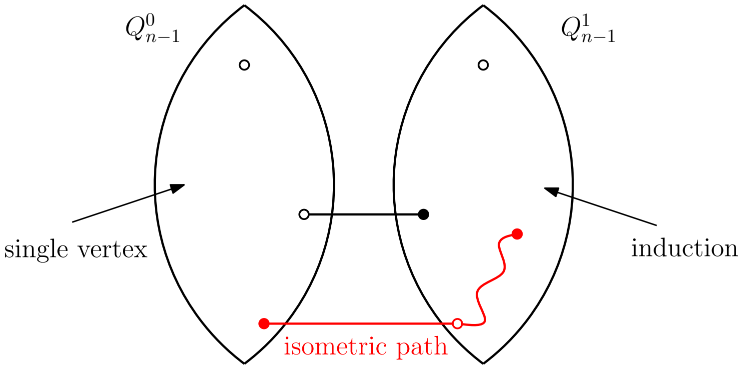

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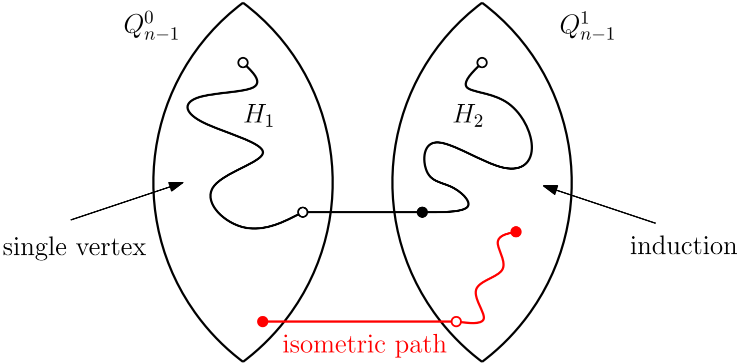

New results

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Let C be an isometric cycle in Q_n of length divisible by four, then for $n \geq 6$ the graph $Q_n - V(C)$ is Hamiltonian laceable.

- up to $2n$ faulty vertices

Theorem:

Let T be a balanced isometric tree in Q_n, then for $n \geq 4$ the graph $Q_n - T$ is Hamiltonian laceable.
Let S be an almost-balanced isometric tree in Q_m, then for $m \geq 5$ the graph $Q_m - S$ is Hamiltonian laceable.

- part of the proof is verified by a computer

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Let S be an almost-balanced isometric tree in Q_m, then for $m \geq 5$ the graph $Q_m - S$ is Hamiltonian laceable.

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Open problem

Conjecture:

Let C be an isometric cycle in Q_n of length not divisible by four, then for $n \geq 6$ the graph $Q_n - V(C)$ is Hamiltonian laceable.

Thank you for your attention.